Nonlinear Free-Surface Effects on a Submerged Sphere

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Nonlinear effects on the wave-induced forces acting on a submerged sphere in a steady translation are investigated. A recent endeavor by Tuck[1] indicates that in the case of a submerged cylinder the forces due to the nonlinearity of the free surface are quite discernible in comparison with the second-order forces derived from the linearized homogeneous free-surface condition. The purpose of the present work is to assess the influence of the nonlinearity upon a three-dimensional water-wave problem.

Nomenclature

$\bar{a}$ = radius of sphere
a = $(g/U^2)\bar{a}$, parameter of Taylor's expansion in vicinity of sphere
c = ratio of radius of sphere to depth of submergence
C_L = wave-lift coefficient
C_R = wave-resistance coefficient
f = $U/(gh)^{1/2}$, Froude number based on depth of submergence
F_n = KG_n, free-surface correction to G_n
g = gravitational acceleration
G_n = WF_{n-1}, body boundary correction to F_{n-1}
$\bar{I}$ = function
I = conjugate function of $\bar{I}$
K = Kotchin's operator
L_{ij} = local terms
M_x = strength of horizontal doublet
M_y = strength of vertical doublet
p_{ij} = coefficient of Taylor's expansion
p_ξ = pressure slope
q_{ij} = coefficient of Taylor's expansion
R_{ij} = radiation terms
s_{ij} = sum of L_{ij} and R_{ij}
$u(\theta)$ = $y + i(x \cos\theta + z \sin\theta)$, complex variable
U = constant forward speed of moving sphere
$v(\theta)$ = $h + i(\xi \cos\theta + \zeta \sin\theta)$, complex variable
W = Weiss' operator
$\bar{x}, \bar{y}, \bar{z}$ = spatial variable
x, y, z = dimensionless spatial variable
α = coefficient of Taylor's expansion
ϵ = expansion parameter of velocity potential
θ = angular variable
μ = $[h^2 + (\xi \cos\theta + \zeta \sin\theta)^2]^{1/2} \sec^2\theta$
ν = g/U^2, wave number
ξ, η, ζ = spatial variable
ρ = density of fluid
τ = arc tan $(\xi \cos\theta + \zeta \sin\theta)/h$
φ = velocity potential

Introduction

THE phenomenon of a wave disturbance generated by a solid body in an arbitrary motion near a free surface is definitely nonlinear, and the exact description of a problem involving such a disturbance is unsurmountably difficult. Therefore, the approximations of successive orders are customarily sought in an asymptotic sense, with an appropriate perturbation parameter for the particular problem. The ratio between the size of the body and the depth of submergence is chosen to be such a parameter in the present case.

Using the linearized free-surface condition, Havelock[2] and Bessho[3] have determined the first-order and higher-order forces encountered by a translating sphere. However, if the nonlinear terms appearing in the Bernoulli equation are retained, for the second-order or higher-order approximation, the free-surface condition becomes inhomogeneous. It can be shown that the inhomogeneous part is composed merely of the known lower-order potentials (see Wehausen and Laitone[4]); hence, the inclusion of the particular solution yields the complete approximation of the nonlinear water-wave problem of the specified order. It can be conceived that the inhomogeneous part corresponds physically to an additional source of wave disturbance on the free surface, which is of nonlinear character.

As might be expected, the expression of the inhomogeneous part ceases to be amenable in the usual perturbation scheme when the order of approximation exceeds the second order. In this work, therefore, a method is presented for applying the second-order, nonlinear correction to the first-order approximation of the water-wave problem arising from a translating sphere near the free surface. The analysis involving the determination of second-order approximations can be greatly simplified if the potential is expressed in terms of a Taylor's series for the small region surrounding the submerged sphere. When the second-order potentials are known, the wave-induced forces, such as the resistance and the lift, can be determined by invoking the Lagally theorem.

General Formulation

With respect to the frame of reference (the Cartesian $\bar{x}$, $\bar{y}$, $\bar{z}$ coordinate axes attached to the center of a submerged sphere), the flow problem of an inviscid, incompressible fluid arising from the forward motion of the sphere with a constant speed U can be viewed as that due to a uniform stream flow past the submerged sphere.

Since the wave disturbance on the free surface depends directly upon the disturbance of the uniform stream by a sphere, the degree of surface deviation from the undisturbed level $\bar{y} = \bar{h}$ can be best measured in terms of a perturbation parameter ϵ involving the radius of the sphere $\bar{a}$ and the depth of submergence $\bar{h}$.

If the parameter ϵ is chosen as the ratio $\bar{a}/\bar{h}$, either in the case of a sufficiently deep submergence or in the case of a sufficiently small sphere, the flow problem can be adequately described by the velocity potential consisting of the first few terms in the expansion

$$\varphi(\bar{x},\bar{y},\bar{z}) = -U\bar{x} + \epsilon\varphi^{(1)} + \epsilon^2\varphi^{(2)} + \dots \quad (1)$$

Here the assumption of irrotationality and incompressibility leads to

$$\nabla^2\varphi^{(i)}(\bar{x},\bar{y},\bar{z}) = 0 \qquad \text{in } -\infty < \bar{y} < \bar{h} \quad (2)$$

and all orders of the potential $\varphi^{(i)}$ must satisfy the kinematic condition on the surface of the sphere $\bar{r} = \bar{a}$, which is

$$\partial\varphi^{(i)}/\partial\bar{r} = V_{\bar{n}} \quad (3)$$

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where $\bar{r}$ is the position vector with respect to the moving coordinates introduced earlier, and V_n is a known function that represents the normal velocity induced by the external singularity.

Furthermore, $\varphi^{(i)}$ must also satisfy the dynamic condition on the free surface $\bar{y} = \bar{h}$; that is,

$$\varphi_{\bar{y}}^{(i)} + (U^2/g)\varphi_{\bar{x}\bar{x}}^{(i)} = (U/g)(p_{\bar{x}}/\rho) \tag{4}$$

where ρ denotes the density of the fluid, g is the gravitational acceleration, and the inhomogeneous part is a functional of the known lower-order potentials

$$p_{\bar{x}}^{(i)} = F_i\{\varphi^{(1)}, \ldots \varphi^{(i-1)}\} \tag{5}$$

Finally, at large distances, the disturbance caused by a submerged sphere should vanish,

$$\nabla\varphi^{(i)}(\bar{x},\bar{y},\bar{z}) = \begin{cases} 0(\bar{x}^{-1/2}) & \text{with } \bar{x} \to +\infty \\ 0 & \text{with } \bar{y} \to -\infty \end{cases} \tag{6}$$

To show clearly the dependence of the solutions associated with each value of the parameter ϵ on the wave number $\nu = g/U^2 = 2\pi/\lambda$, where λ denotes the wavelength, we shall make the space variables and other relevant variables dimensionless; that is,

$$x = \nu\bar{x} \qquad y = \nu\bar{y} \qquad z = \nu\bar{z}$$
$$a = \nu\bar{a} \qquad \text{and} \qquad h = \nu\bar{h} \tag{7}$$

Here, in order for the approximation to approach the exact solution in an asymptotic sense, the magnitude of $\epsilon = a/h$ should remain small. Therefore, following Tuck's notion, let us consider the dimensionless quality a itself to be a small parameter with $h = 0(1)$. Then, in terms of the dimensionless variables, the boundary-value problem described by Eqs. (2–4 and 6) becomes

$$\nabla^2\varphi^{(i)}(x,y,z) = 0 \qquad \text{in } -\infty < y < h \tag{8a}$$

$$\partial\varphi^{(i)}/\partial r = V_n \qquad \text{on } r = a \tag{8b}$$

$$\varphi_y^{(i)} + \varphi_{xx}^{(i)} = (U/g)(p_x^{(i)}/\rho) \qquad \text{on } y = h \tag{8c}$$

$$\nabla\varphi^{(i)}(x,y,z) = \begin{cases} 0(x^{-1/2}) & \text{with } x \to +\infty \\ 0 & \text{with } y \to -\infty \end{cases} \tag{8d}$$

As a consequence of retaining the nonlinear terms appearing in the Bernoulli equation, we have, specifically,

$$(1/\rho)p_x^{(1)} = 0$$

and (9)

$$\frac{1}{\rho}p_x^{(2)} = \frac{\partial}{\partial x}(\nabla\varphi^{(1)})^2 - \varphi_x^{(1)}\frac{\partial}{\partial y}(\varphi_y^{(1)} + \varphi_{xx}^{(1)})$$

Successive Approximation

The first-order solution of the boundary-value problem (8) with V_n corresponding to the flow from a source can be found in the existing literature.[4] As in uniform flow, a sphere is represented by a doublet of strength $M = -Ua^3/2$, so the first-order solution to our boundary-value problem could have been derived from x-differentiation. However, we prefer the following equivalent expression given by Kotchin[5] in terms of a variable $u(\theta) = y + i(x\cos\theta + z\sin\theta)$:

$$\bar{\varphi}^{(1)} = M\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta}{[u(\theta)]^2}d\theta + M\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta}{[\tau(\theta)]^2}d\theta + MRe\left\{-\frac{i}{\pi}\int_{-\pi}^{\pi}\frac{d}{d\tau}[e^{\tau\sec^2\theta}\operatorname{Ei}(-\tau\sec^2\theta)]\sec\theta d\theta + 2\int_{-\pi/2}^{\pi/2}e^{\tau\sec^2\theta}\sec^3\theta d\theta\right\} \tag{10}$$

where θ denotes the wave angle, $\tau(\theta) = u(\theta) - 2h$, and $\operatorname{Ei}(\zeta)$ represents the principal value of the exponential integral

$$\operatorname{Ei}^{\pm}(\zeta) = \int_{-\infty\pm io}^{\zeta} e^t\frac{dt}{t} = \mp i\pi + \operatorname{Ei}(\zeta) \tag{11}$$

which is defined in the complex t plane cut along the negative real axis.

The integral appearing in the first term of (10) has the identity

$$\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta}{[u(\theta)]^2}d\theta = \frac{x}{R^3} \qquad \text{where } R^2 = x^2 + y^2 + z^2 \tag{12}$$

Therefore it corresponds to the velocity potential of a horizontal doublet of unit strength, and the potential exactly satisfies the kinematic condition over the surface of the sphere when the uniform stream of the potential x is only an external singularity. Observing that the application of the Weiss sphere theorem to the potential x should yield the desired result with the exception of the additive factor $a^3/2$, one may introduce an operator W, which transforms a harmonic function in the space below the undisturbed free surface into a function that is harmonic everywhere outside a sphere, with satisfactory decay at infinity; and one denotes

$$W[x] = \frac{a^3}{2}\left[\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta}{[u(\theta)]^2}d\theta\right] \tag{13}$$

The three integrals in the second term of (10) represent the free-surface image of the horizontal doublet; this image is obtained by satisfying the homogeneous free-surface condition and the proper radiation condition. Thus, one may introduce an operator K, which transforms a harmonic function outside a sphere into a function that is harmonic everywhere in the space below the undisturbed free surface and satisfies the linearized free-surface condition and the radiation condition; and write

$$K\left[\frac{x}{R^3}\right] = \frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta}{[\tau(\theta)]^2}d\theta - \frac{i}{\pi}\int_{-\pi}^{\pi}\frac{d}{d\tau}\times [e^{\tau\sec^2\theta}\operatorname{Ei}(-\tau\sec^2\theta)]\sec\theta\, d\theta + 2\int_{-\pi/2}^{\pi/2}e^{\tau\sec^2\theta}\sec^3\theta d\theta \tag{14}$$

Thus, in terms of the operators W and K, the first approximation can be written in the form

$$\bar{\varphi}^{(1)} = -URe\{W[x] + KW[x]\} \tag{15}$$

Wehausen[4] indicates that a tactful way to evaluate the higher-order approximations for the wholly linearized problem is to follow the sequence of operations represented by

$$\begin{array}{lll} G_0 = 0 & F_0 = x & \bar{\varphi}^{(0)} = -Ux \\ G_1 = WF_0 & F_1 = KG_1 & \bar{\varphi}^{(1)} = -URe[WF_0 + KG_1] \\ G_2 = WF_1 & F_2 = KG_2 & \bar{\varphi}^{(2)} = -URe[WF_1 + KG_2] \\ \ldots & \ldots & \ldots \qquad \ldots \\ G_n = WF_{n-1} & F_n = KG_n & \bar{\varphi}^{(n)} = -URe[WF_{n-1} + KG_n] \end{array} \tag{16}$$

where, for a sufficiently small value of the perturbation parameter $\epsilon = a/h$, with $h = 0(1)$, the $\bar{\varphi}^{(n)}$ are of decreasing order and the sequence

$$\bar{\varphi} = \sum_{n=0}^{\infty}\epsilon^n\bar{\varphi}^{(n)}$$

converges.

For the inhomogeneous problem, one may, in accordance with a procedure adopted by Tuck,[1] modify the scheme (16) by adding to $\bar{\varphi}^{(i)}$ the particular solution of the equation describing the inhomogeneous free-surface condition. One then obtains the complete approximation of the ith-order inhomogeneous problem,

$$\varphi^{(i)} = -URe[WF_{i-1} + KG_i] + \varphi_p^{(i)} \quad (17)$$

where the solution $\varphi_p^{(i)}$ due to a pressure distribution $p^{(i)}(x,z)$ on the free surface is given by Wehausen in the form

$$\varphi_p^{(i)} = \frac{1}{4\pi U}\int_{-\infty}^{\infty} \frac{p_\xi^{(i)}(\xi,\zeta)}{\rho} H(x,y,z;\xi, -h,\zeta)d\xi d\zeta \quad (18)$$

where

$$H = Re\left\{\frac{1}{\pi}\int_{-\pi}^{\pi} e^{t\sec^2\theta}\,\mathrm{Ei}(-t\sec^2\theta)\sec^2\theta d\theta + i2\int_{-\pi/2}^{\pi/2} e^{t\sec^2\theta}\sec^2\theta d\theta\right\}$$

and

$$t(\theta) = u(\theta) - v(\theta) = u(\theta) - [h + i(\xi\cos\theta + \zeta\sin\theta)]$$

Expansion with Parameter a in the Vicinity of a Sphere

All orders of the potential F_i associated with the approximation scheme (16) are harmonic everywhere beneath the free surface. Therefore, one may expand F_i in a Taylor series in the neighborhood of the submerged sphere. As will be seen presently, such an expansion, based on the assumption that a itself is a small parameter not only facilitates the evaluations of the W and K transformations in each stage of the approximation, but also enables one to determine the wave-induced forces on the sphere quite readily, by application of the Lagally theorem.

The potential $F_1 = KG_1$ [given by (14)] corresponds to the free-surface image of a horizontal doublet. The expansion of this potential in a Taylor series about the origin yields

$$K\left[\frac{x}{R^3}\right] = \frac{2}{a^3}F_1 = \sum_{n=0}^{\infty}\frac{1}{n!}\left\{\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{(n+1)!}{(2h)^{n+2}}\times u^n(\theta)\cos\theta d\theta - \frac{i}{\pi}\int_{-\pi}^{\pi}\frac{d^{n+1}}{d\tau^{n+1}}[e^{\tau\sec^2\theta}\,\mathrm{Ei}(-\tau\sec^2\theta)]|_{\tau=-2h}\times u^n(\theta)\sec\theta d\theta + 2\int_{-\pi/2}^{\pi/2} e^{-2h\sec^2\theta}u^n(\theta)\sec^{2n+3}\theta d\theta\right\} \quad (19)$$

where

$$u(\theta) = y + i(x\cos\theta + z\sin\theta)$$

Using the mathematical identities (see Kim[6]), the first few terms of the real part of the expansion (19) can be expressed as

$$Re\{K[x/R^3]\} = Re[(2/a^3)F_1] = p_{02} - p_{11}x + p_{12}y - p_{21}xy + \tfrac{1}{2}p_{12}(z^2 - x^2) + \tfrac{1}{2}p_{22}(y^2 - z^2) + 0[u^3(\theta)] \quad (20)$$

where

$$p_{02} = e^{-h}[K_0(h) + K_1(h)]$$

$$p_{11} = \frac{1}{(2h)^3} - \frac{4}{\pi}\int_0^{\pi/2}\left[e^{-2h\sec^2\theta}\,\mathrm{Ei}(2h)\sec^6\theta - \sum_{k=1}^{2}\frac{(k-1)!}{(2h)^k}\cos^{2k-4}\theta\right]d\theta$$

$$p_{12} = e^{-h}[K_0(h) + (1 + 1/2h)K_1(h)]$$

$$p_{21} = \frac{3}{(2h)^4} - \frac{4}{\pi}\int_0^{\pi/2}\left[e^{-2h\sec^2\theta}\,\mathrm{Ei}(2h\sec^2\theta)\sec^6\theta - \sum_{k=1}^{3}\frac{(k-1)!}{(2h)^k}\cos^{2k-6}\theta\right]d\theta$$

$$p_{22} = e^{-h}[(1 + 1/4h)K_0(h) + (1 + 3/4h + 1/2h^2)K_1(h)] \quad (21)$$

where $Kn(h)$ denotes the nth-order modified Bessel function of the second kind.

The significance of the expression (20) is that, to the first order in terms of distance from the center, the free-surface image of the horizontal doublet induces a disturbance about the submerged sphere in the form of horizontal and vertical uniform streams. In the next stage of the approximation, therefore, one may, instead of determining the full transformation, simply repair the boundary condition by canceling only such first-order disturbances.

Ignoring the constant term in (20) thus, one may write

$$Re[F_1] = \frac{a^3}{2}Re\left\{K\left[\frac{x}{R^3}\right]\right\} = -\frac{a^3}{2}(p_{11}x - p_{12}y) + 0[a^3u^2(\theta)] \quad (22)$$

where the neglected term is $0(a^5)$ as long as $u(\theta) = 0(a)$. Now the application of the W transformation yields

$$Re[G_2] = -\frac{a^3}{2}\left[p_{11}\left(\frac{a^3}{2}\frac{x}{R^3}\right) - p_{12}\left(\frac{a^3}{2}\frac{y}{R^3}\right)\right] + 0\left[a^8\frac{u^2}{R^5}\right] \quad (23)$$

and then, by the K transformation, one finds

$$Re[F_2] = -\left(\frac{a^3}{2}\right)^2 Re\left\{p_{11}K\left[\frac{x}{R^3}\right] - p_{12}K\left[\frac{y}{R^3}\right]\right\} + 0\left\{a^8K\left[\frac{u^2}{R^5}\right]\right\} \quad (24)$$

where

$$K\left[\frac{y}{R^3}\right] = \sum_{n=0}^{\infty}\frac{1}{n!}\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(n+1)!}{(2h)^{n+2}}u^n(\theta)d\theta + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{d^{n+1}}{d\tau^{n+1}}[e^{\tau\sec^2\theta}\,\mathrm{Ei}(-\tau\sec^2\theta)]|_{\tau=-2h}\times u^n(\theta)\sec^2\theta d\theta + 2i\int_{-\pi/2}^{\pi/2} e^{-2h\sec^2\theta}u^n(\theta)\sec^{2n+4}\theta d\theta\right\} \quad (25)$$

Hence, the first few terms of the real part of this expansion are found to be

$$Re\{K[y/R^3]\} = -q_{01} - p_{12}x - q_{11}y - \tfrac{1}{2}q_{21}y^2 + \tfrac{1}{2}p_{21}x^2 + \tfrac{1}{2}q_{22}z^2 - xyp_{22} + 0[u^3(\theta)] \quad (26)$$

where

$$q_{01} = \frac{1}{(2h)^2} - \frac{4}{\pi}\int_0^{\pi/2}\left[e^{-2h\sec^2\theta}\,\mathrm{Ei}(2h\sec^2\theta) - \frac{1}{2h}\right]\sec^2\theta d\theta$$

$$q_{11} = \frac{2}{(2h)^3} - \frac{4}{\pi}\int_0^{\pi/2}\left[e^{-2h\sec^2\theta}\,\mathrm{Ei}(2h\sec^2\theta)\sec^4\theta - \sum_{k=1}^{2}\frac{(k-1)!}{(2h)^k}\cos^{2k-4}\theta\right]\sec^2\theta d\theta$$

$$q_{21} = \frac{6}{(2h)^4} - \frac{4}{\pi}\int_0^{\pi/2}\left[e^{-2h\sec^2\theta}\,\mathrm{Ei}(2h\sec^2\theta)\sec^6\theta - \sum_{k=1}^{3}\frac{(k-1)!}{(2h)^k}\cos^{2k-6}\theta\right]\sec^2\theta d\theta$$

$$q_{22} = \frac{3}{(2h)^4} - \frac{4}{\pi}\left[e^{-2h\sec^2\theta}\,\mathrm{Ei}(2h\sec^2\theta)\sec^6\theta - \sum_{k=1}^{3}\frac{(k-1)!}{(2h)^k}\cos^{2k-6}\theta\right]\tan^2\theta d\theta \quad (27)$$

Finally, substituting (20) and (26) into (24), the free-surface image of G_2 can be written explicitly as

$$Re[F_2] = (a^3/2)^2\{p_{11}[p_{11}x - p_{12}y + p_{21}xy - \tfrac{1}{2}p_{12}(z^2 - x^2) - \tfrac{1}{2}p_{22}(y^2 - z^2)] - p_{12}[p_{12}x + q_{11}y + \tfrac{1}{2}q_{12}y^2 - \tfrac{1}{2}p_{12}x^2 - \tfrac{1}{2}q_{22}z^2 + xyp_{22}]\} + 0[a^8u(\theta)] \quad (28)$$

where the neglected term is $0(a^9)$.

Nonlinear Correction

Since the boundary condition on the free surface is inhomogeneous because of the presence of nonlinear terms in the Bernoulli equation, all orders of the solution of the boundary-value problem (8) comprise the general solution $\bar{\varphi}^{(i)}$ and the particular solution $\varphi_p^{(i)}$. Thus, by use of expansions (22, 23, and 28), the first- and second-order solutions of the problem can be written as

$$\varphi^{(1)} = -URe[G_1 + F_1] + \varphi_p^{(1)} = -U\left[\left(\frac{a^3}{2}\right)\left(\frac{x}{R^3} - p_{11}x + p_{12}y\right) + 0(a^5)\right] + 0 \quad (29)$$

and

$$\varphi^{(2)} = -URe[G_2 + F_2] + \varphi_p^{(2)} = U\left\{\left(\frac{a^3}{2}\right)^2\left[p_{11}\frac{x}{R^3} - p_{11}^2x + p_{11}p_{12}y + \ldots - p_{12}y + p_{12}^2x + p_{12}q_{11}y + \ldots + p_{12}p_{22}xy\right] + 0(a^9)\right\} + \varphi_p^{(2)} \quad (30)$$

where, in terms of the small parameter a, the particular solution $\varphi_p^{(2)}$ given by (18) can now be expressed as

$$\varphi_p^{(2)} = \left(\frac{a^3}{2}\right)^2\sum_{n=0}^{\infty}\alpha_n = \frac{1}{4\pi U}\left(\frac{a^3}{2}\right)^2\int_{-\infty}^{\infty}\!\!\int\frac{p_\xi^{(2)}}{\rho(a^3/2)^2}\cdot \mathrm{Re}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\pi}\int_{-\pi}^{\pi}\frac{d^n}{dv^n}e^{-v\sec^2\theta}\,\mathrm{Ei}(v\sec^2\theta)u^n(\theta)\times \sec\theta d\theta + \frac{i2}{n!}\int_{-\pi/2}^{\pi/2}e^{-v\sec^2\theta}\,u^n(\theta)\sec^{2n+2}\theta d\theta\right\}d\xi d\zeta \quad (31)$$

with u and v denoting $u(\theta) = y + i(x\cos\theta + z\sin\theta)$ and $v(\theta) = h + i(\xi\cos\theta + \zeta\sin\theta)$, respectively.

The only term necessary for the calculation of the hydrodynamic force on doublets is simply

$$\alpha_2 = \frac{1}{4\pi U}\int_{-\infty}^{\infty}\!\!\int\frac{p_\xi^{(2)}}{\rho(a^3/2)^2}Re\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d^2}{dv^2}\times e^{-v\sec^2\theta}\,\mathrm{Ei}\,(v\sec^2\theta)u^2(\theta)\sec^2\theta d\theta + i\int_{-\pi/2}^{\pi/2}e^{-v\sec^2\theta}\,u^2(\theta)\sec^6\theta d\theta\right\}d\xi d\zeta \quad (32)$$

However, this expression can be reduced to a more tractable form as

$$\alpha_2 = \frac{1}{4\pi U}\int_{-\infty}^{\infty}\!\!\int\frac{p_\xi^{(2)}}{\rho(a^3/2)^2}Re\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[e^{-v\sec^2\theta}\,\mathrm{Ei}\times(v\sec^2\theta)\sec^4\theta - \sum_{k=1}^{2}\frac{(k-1)!}{[v(\theta)]^k}\cos^{2k-4}\theta\right]T(x,y,z;\theta)\sec^2\theta d\theta + i\int_{-\pi/2}^{\pi/2}e^{-v\sec^2\theta}\,T(x,y,z;\theta)\sec^6\theta d\theta\right\}d\xi d\zeta \quad (33)$$

where $T(x,y,z;\theta) = y^2 - (x\cos\theta + z\sin\theta)^2 + i2y(x\cos\theta + z\sin\theta)$.

In the three-dimensional problem, the pressure-slope $p_\xi^{(2)}/\rho$ assumes a formidable form; that is,

$$(1/\rho)p_\xi^{(2)}(\xi,0,\zeta;0,-h,0) = (\partial/\partial\xi)(\nabla\varphi^{(1)})^2 - \varphi_\xi^{(1)}(\partial/\partial\eta)(\varphi_\eta^{(1)} + \varphi_{\xi\xi}^{(1)}) = 2[\varphi_\xi^{(1)}\varphi_{\xi\xi}^{(1)} + \varphi_\eta^{(1)}\varphi_{\xi\eta}^{(1)} + \varphi_\zeta^{(1)}\varphi_{\xi\zeta}^{(1)}] - \varphi_\xi^{(1)}[\varphi_{\eta\eta}^{(1)} + \varphi_{\eta\xi\xi}^{(1)}] \quad (34)$$

where the potential $\varphi^{(1)}$ represents a doublet of the form written below [which is just the first-order approximation of our problem (10)]:

$$\varphi^{(1)} = \varphi^{(1,1)}(\xi,0,\zeta;0,-h,0) + \varphi^{(1,2)}(\xi,0,\zeta;0,-h,0) \quad (35)$$

where

$$\varphi^{(1,1)}(\xi,0,\zeta;0,-h,0) = \frac{M}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{i\cos\theta}{[v(\theta)]^2} + \frac{i\cos\theta}{[\bar{v}(\theta)]^2}\right\}d\theta$$

$$\varphi^{(1,2)}(\xi,0,\zeta;0,-h,0) = MRe\times\left\{-\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{d}{d\bar{v}}[e^{\bar{v}\sec^2\theta}\,\mathrm{Ei}(-\bar{v}\sec^2\theta)]\sec\theta d\theta + 2\int_{-\pi/2}^{\pi/2}e^{\bar{v}\sec^2\theta}\sec^3\theta d\theta\right\}$$

with M denoting $-(U/2)a^3$, $v(\theta) = h + i(\xi\cos\theta + \zeta\sin\theta)$, and $\bar{v}(\theta) = -h + i(\xi\cos\theta + \zeta\sin\theta)$. Except for the η-derivative of odd order, all other partial derivatives of $\varphi^{(1,1)}$ vanish identically; hence,

$$(2/M^2)[\varphi_\xi^{(1,1)}\varphi_{\xi\xi}^{(1,1)} + \varphi_\eta^{(1,1)}\varphi_{\xi\eta}^{(1,1)} + \varphi_\zeta^{(1,1)}\varphi_{\xi\zeta}^{(1,1)}] - (1/M^2)\varphi_\xi^{(1,1)}[\varphi_{\eta\eta}^{(1,1)} + \varphi_{\eta\xi\xi}^{(1,1)}] = (2/M^2)[\varphi_\eta^{(1,1)}\varphi_{\xi\eta}^{(1,1)}] = 2r_1r_2 \quad (36)$$

with

$$r_1 = -\frac{6\xi h}{(R')^5} \qquad r_2 = -\frac{6h}{(R')^5}\left[1 - 5\left(\frac{\xi}{R'}\right)^2\right]$$

and $(R')^2 = h^2 + \xi^2 + \zeta^2$

Next, using the remaining terms in (35), one may write

$$(2/M^2)[\varphi_\xi^{(1,2)}\varphi_{\xi\xi}^{(1,2)} + \varphi_\eta^{(1,2)}\varphi_{\xi\eta}^{(1,2)} + \varphi_\zeta^{(1,2)}\varphi_{\xi\zeta}^{(1,2)}] - (1/M^2)\varphi_\xi^{(1,2)}[\varphi_{\eta\eta}^{(1,2)} + \varphi_{\eta\xi\xi}^{(1,2)}] = 2(s_{11}s_{21} + s_{12}s_{22} + s_{13}s_{23}) - s_3s_4 \quad (37)$$

with

$$s_1 = s_{11} + s_{12} + s_{13} = Re\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}\left[e^{\bar{v}\sec^2\theta}\,\mathrm{Ei}(-v\sec^2\theta)\times\sec^4\theta - \sum_{k=1}^{2}\frac{(-1)^k(k-1)!}{[\bar{v}(\theta)]^k}\cos^{2k-4}\theta\right]\sec\theta(\cos\theta - i + \sin\theta)d\theta + 2i\int_{-\pi/2}^{\pi/2}e^{\bar{v}\sec^2\theta}\sec^5\theta(\cos\theta - i + \sin\theta)d\theta\right\}$$

$$s_2 = s_{21} + s_{22} + s_{23} = Re\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}\left[e^{\bar{v}\sec^2\theta}\,\mathrm{Ei}(-v\sec^2\theta)\times\sec^6\theta - \sum_{k=1}^{3}\frac{(-1)^k(k-1)!}{[\bar{v}(\theta)]^k}\cos^{2k-6}\theta\right](i\cos\theta + 1 + i\sin\theta)d\theta + 2i\int_{-\pi/2}^{\pi/2}e^{\bar{v}\sec^2\theta}\sec^6\theta(i\cos\theta + 1 + i\sin\theta)d\theta\right\}$$

$$s_3 = s_{11}$$

and

$$s_4 = Re\left\{\frac{i}{\pi}\int_{-\pi}^{\pi}\left[\sum_{k=1}^{3}\frac{(-1)^k(k-1)!}{[v(\theta)]^k}\cos^{2k-6}\theta - \cos^2\theta\sum_{k=1}^{4}\frac{(-1)^k(k-1)!}{[\bar{v}(\theta)]^k}\cos^{2k-8}\theta\right]\sec\theta d\theta\right\} = Re\left\{-\frac{i}{\pi}\int_{-\pi}^{\pi}\frac{3!}{[\bar{v}(\theta)]^4}\cos\theta d\theta\right\}$$

where $\bar{t}(\theta) = (\eta + q) + i(\xi - p)\cos\theta + i(\zeta - r)\sin\theta$.

From (36) and (37), the pressure term (34) can be expressed explicitly as

$$\frac{p_\xi^{(2)}}{\rho(a^3/2)^2} = U^2\{2[s_{11}s_{21} + (r_1 + s_{12})(r_2 + s_{22}) + s_{13}s_{23}] - s_{11}s_4\} \quad (38)$$

Therefore, the expression of α_2 given by (33) becomes

$$\frac{\alpha_2}{U} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int \{2[s_{11}s_{21} + (r_1 + s_{12})(r_2 + s_{22}) + s_{13}s_{23}] - s_{11}s_4\}\cdot Re\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[e^{-v\sec^2\theta}\,\mathrm{Ei}(v\sec^2\theta)\sec^4\theta - \sum_{k=1}^{2}\frac{(k-1)!}{[v(\theta)]^k}\cos^{2k-4}\theta\right]T(x,y,z;\theta)\sec^2\theta d\theta + i\int_{-\pi/2}^{\pi/2} e^{-v\sec^2\theta}\,T(x,y,z;\theta)\sec^6\theta d\theta\right\} d\xi d\zeta \quad (39)$$

and it follows from (31) and (39) that

$$\varphi_p^{(2)} = -U(a^3/2)^2(-\alpha_1/U - \alpha_2/U) + 0(a^9) \quad (40)$$

Wave-Induced Resistance and Lift

As soon as the first- and second-order solutions are known, wave-induced dynamic forces (such as the resistance and the lift acting on a submerged sphere at the depth h) can be most easily determined by application of Lagally's theorem, which states that when the external singularities consist of doublets with strength components $\tilde{M}$, the forces acting on the submerged body are given by

$$\tilde{F} = -4\pi\rho\Sigma(\tilde{M}\cdot\nabla)\tilde{q} \quad (41)$$

where $\tilde{q}$ is the velocity due to all external effects at the location of the singularity.

The external singularities in the present problem consist of one first-order doublet of the strength component

$$M_x^{(1)} = -(U/2)a^3 \quad (42)$$

and two second-order doublets of the strength components

$$M_x^{(2)} = -(U/4)a^6(-p_{11}) \quad \text{and}$$

$$M_y^{(2)} = -(U/4)a^6 p_{12} \quad (43)$$

Furthermore, from (29–31), the velocity potential describing the flow around the submerged sphere can be expressed as

$$\varphi(x,y,z) = -Ux + M_x^{(1)}\left\{\frac{x}{R^3} + Re\left[K\left(\frac{x}{R^3}\right)\right]\right\} + M_x^{(2)}\left\{\frac{x}{R^3} + Re\left[K\left(\frac{x}{R^3}\right)\right]\right\} + M_y^{(2)}\left\{\frac{y}{R^3} + Re\left[K\left(\frac{y}{R^3}\right)\right]\right\} + \left(-\frac{U}{4}a^6\right)\sum_0^\infty\left(-\frac{\alpha_n}{U}\right) =$$
$$-Ux - \frac{U}{2}a^3\frac{x}{R^3} - \frac{U}{2}a^3\left(\ldots - p_{21}xy - \frac{1}{2}p_{12}x^2 + \frac{1}{2}p_{22}y^2 + \ldots\right) + \sum_3^\infty 0(a^3)u^n + \frac{U}{4}a^6p_{11}\frac{x}{R^3} + \frac{U}{4}a^6p_{11}\left(\ldots - p_{21}xy - \frac{1}{2}p_{12}x^2 + \frac{1}{2}p_{22}y^2 + \ldots\right) - \frac{U}{4}a^6p_{12}\frac{y}{R^3} - \frac{U}{4}a^6p_{12}\left(\ldots + \frac{1}{2}p_{21}x^2 - p_{22}xy + \ldots\right) + \frac{U}{4}a^6\left[\ldots + \frac{1}{U}\alpha_2(x^2,y^2,z^2,xy,yz,zx)\right] + \sum_3^\infty 0(a^6)u^n \quad (44)$$

Therefore, the x- and y-component forces are

$$F_x \equiv R = -4\pi\rho\left[(M_x^{(1)} + M_x^{(2)})\frac{\partial}{\partial x} + M_y^{(2)}\frac{\partial}{\partial y}\right]\times\frac{\partial}{\partial x}\varphi = -4\pi\rho M_x^{(1)}\left(M_x^{(1)}\frac{\partial^2}{\partial x^2}Re\left[K\left(\frac{x}{R^3}\right)\right] + 2M_x^{(2)}\frac{\partial^2}{\partial x^2}Re\left[K\left(\frac{x}{R^3}\right)\right] + M_y^{(2)}\left\{\frac{\partial^2}{\partial x^2}Re\left[K\left(\frac{x}{R^3}\right)\right] + \frac{\partial^2}{\partial y\partial x}Re\left[K\left(\frac{x}{R^3}\right)\right]\right\} + \frac{U}{4}a^6\frac{\partial^2}{\partial x^2}\left(\frac{\alpha_2}{U}\right) + 0(a^9)\right) \quad (45)$$

and

$$F_y \equiv L = -4\pi\rho\left[(M_x^{(1)} + M_x^{(2)})\frac{\partial}{\partial x} + M_y^{(2)}\frac{\partial}{\partial y}\right]\times\frac{\partial}{\partial y}\varphi = -4\pi\rho M_x^{(1)}\left(M_x^{(1)}\frac{\partial^2}{\partial x\partial y}Re\left[K\left(\frac{x}{R^3}\right)\right] + 2M_x^{(2)}\frac{\partial^2}{\partial y\partial x}Re\left(\frac{x}{R^3}\right) + M_y^{(2)}\left\{\frac{\partial^2}{\partial x\partial y}Re\left[K\left(\frac{y}{R^3}\right)\right] + \frac{\partial^2}{\partial y^2}Re\left(\frac{x}{R^3}\right)\right\} + \frac{U}{4}a^6\frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha_2}{U}\right) + 0(a^9)\right) \quad (46)$$

Dividing the forces by the buoyant force of the sphere—that is,

$$\tfrac{4}{3}\pi\rho g a^3 = \tfrac{4}{3}\pi\rho U^2 a^3$$

(since in our units $U^2 = g$)—one may define the dimensionless wave-resistance and wave-lift coefficients by

$$C_R = \frac{3R}{4\pi\rho U^2a^3} = \frac{3}{4}a^3\left[p_{12} - a^3p_{12}p_{11} + \frac{a^3}{2}\frac{\alpha_{21}}{U}\right]^* \quad (47)$$

and

$$C_L = \frac{3L}{4\pi\rho U^2a^3} = \frac{3}{4}a^3\left[p_{21} - a^3p_{21}p_{11} + \frac{a^3}{2}\frac{\alpha_{22}}{U}\right]^* \quad (48)$$

Where the asterisk notation is used, the x- and y-component forces, R and L, have the dimension not in pounds but pounds per square feet. Therefore, in order to obtain the actual resistance and lift forces $\bar{R}$ and $\bar{L}$ in pounds, one has to multiply the coefficients C_R and C_L by the actual buoyant force in pounds, i.e.,

$$\bar{R} = \tfrac{4}{3}C_R\pi\rho g\bar{a}^3 \qquad \text{and} \qquad \bar{L} = \tfrac{4}{3}C_L\pi\rho g\bar{a}^3$$

where

$$\frac{\alpha_{21}}{U} \equiv \frac{\partial^2}{\partial x^2}\left(\frac{\alpha_2}{U}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int\{2[s_{11}s_{21} + (r_1 + s_{12})(r_2 + s_{22}) + s_{13}s_{23}] - s_{11}s_4\}\cdot Re\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[e^{-v\sec^2\theta}\,\mathrm{Ei}(v\sec^2\theta)\sec^4\theta - \sum_{k=1}^{2}\frac{(k-1)!}{[v(\theta)]^k}\cos^{2k-4}\theta\right]d\theta + i\int_{-\pi/2}^{\pi/2}e^{-v\sec^2\theta}\sec^4\theta d\theta\right\} d\xi d\zeta \quad (49)$$

It may be noted that the factor expressed by the last two lines of (49) is just equal to $s_{11}(-\xi,\zeta)$, and

$$\frac{\alpha_{22}}{U} \equiv \frac{\partial^2}{\partial x\partial y}\left(\frac{\alpha_2}{U}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int\{[s_1s_{21} + (r_1 + s_{12})(r_2 + s_{22}) + s_{13}s_{23}] - s_{11}s_4\}\cdot Re\left\{\frac{i}{2\pi}\int_{-\pi}^{\pi}\left[e^{-v\sec^2\theta}\,\mathrm{Ei}(v\sec^2\theta)\sec^5\theta - \sum_{k=1}^{2}\frac{(k-1)!}{[v(\theta)]^k}\cos^{2k-5}\theta\right]d\theta - \int_{-\pi/2}^{\pi/2}e^{-v\sec^2\theta}\sec^5\theta d\theta\right\} d\xi d\zeta \quad (50)$$

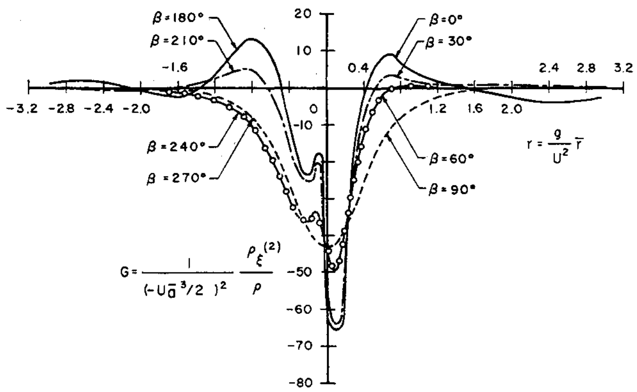

Fig. 1 Dimensionless pressure-slope at various points on free surface.

In Eqs. (47) and (48), the first term indicates the result of the first approximation and terms up to the second term indicate the result of the second approximation, for the linear problem. The nonlinear contribution is represented by the term α_{2i}/U $(i = 1,2)$.

In order to show the dependence of the coefficients C_R and C_L on the forward speed of a submerged sphere U, one may introduce a dimensionless parameter, the Froude number defined by

$$f = U/(gh)^{1/2} \tag{51}$$

Then one obtains $h = 1/f^2$ from the definition of the length units, and when $a/h = c$, one has

$$a = ch = c/f^2 \tag{52}$$

Numerical Procedure

For a certain range of the Froude number f, one has to evaluate the wave-resistance and wave-lift coefficients that are given by (47) and (48), respectively. These coefficients for given h depend upon the size of the disturbance only in powers of a^3. Therefore, when the value of the ratio a/h is equal to c, one may replace the parameter a in formulas (47) and (48) by ch where $h = 1/f^2$. In order to assess a severe disturbance, in the present work the radius-submergence ratio is taken equal to $c = \frac{1}{2}$, and the range of the Froude number $f = 0.4 \sim 1.2$ is considered.

From (47) and (48), one gets

$$C_R = \tfrac{3}{4}(ch)^3[p_{12} - (ch)^3 p_{12} p_{11} + \tfrac{1}{2}(ch)^3(\alpha_{21}/U)] \tag{53}$$

and

$$C_L = \tfrac{3}{4}(ch)^3[p_{21} - (ch)^3 p_{21} p_{11} + \tfrac{1}{2}(ch)^3(\alpha_{22}/U)] \tag{54}$$

where the coefficients p_{11}, p_{12}, and p_{21} are given in (21).

The integrand in the coefficients p_{11} and p_{21} is a function of a product $2h\sec^2\theta$ which increases with the integral variable θ. Therefore, in the evaluation of these integrals, one may use a power series expansion for small values of the product and use an asymptotic expansion for large values of the product. Thus, one obtains

$$e^{-2h\sec^2\theta}\,\mathrm{Ei}(2h\sec^2\theta) - \sum_{k=1}^{n}\frac{(k-1)!}{(2h\sec^2\theta)^k} = \begin{cases} e^{-2h\sec^2\theta}\left[\gamma + \ln(2h\sec^2\theta) + \sum_{k=1}^{\infty}\frac{(2h\sec^2\theta)^k}{kk!} - \sum_{k=1}^{n}\frac{(k-1)!}{(2h\sec^2\theta)^k}\right] & \text{for } 2h\sec^2\theta < M \\ \text{or} \quad \sum_{k=n+1}^{N}\frac{(k-1)!}{(2h\sec^2\theta)^k} & \text{for } 2h\sec^2\theta \geq M \end{cases} \tag{55}$$

where γ denotes Euler's constant, and M is an arbitrary large number $1 < M \leq 10$. Since the asymptotic series becomes divergent if the number of terms are increased indefinitely, N should be taken equal to the largest integer smaller than $2h\sec^2\theta$.

Further, from (49) and (50), one finds the nonlinear contribution to the wave-resistance and wave-lift coefficients as

$$\frac{\alpha_{21}}{U} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int \{2[s_{11}s_{21} + (r_1 + s_{12})(r_2 + s_{22}) + s_{13}s_{23}] - s_{11}s_4\}\, s_5\, d\xi d\zeta \tag{56}$$

and

$$\frac{\alpha_{22}}{U} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int \{2[s_{11}s_{21} + (r_1 + s_{12})(r_2 + s_{22}) + s_{13}s_{23}] - s_{11}s_4\}\, s_6\, d\xi d\zeta \tag{57}$$

Now by use of new quantities μ and τ defined by

$$\mu e^{i\tau} = h\sec^2\theta + i(\xi + \zeta\tan\theta)\sec\theta \tag{58}$$

the terms with s-designation can be resolved into two parts, namely, the local terms L's and the radiation terms R's (one takes these terminologies for convenience, even though the term called local contains the same pattern of decay at infinity as that called radiation). Thus, in compact notation one may define

$$s_{11} + s_{12} + s_{13} = (L_{11} + L_{12} + L_{13}) + (R_{11} + R_{12} + R_{13}) = Re\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\bar{I}(\theta) - \sum_{k=1}^{2}\frac{(k-1)!}{[\mu e^{-i\tau}]^k}\right]\sec^5\theta(\cos\theta - i + \sin\theta)d\theta\right\} + Re\left\{2i\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\, e^{i\mu\sin\tau}\sec^5\theta \times (\cos\theta - i + \sin\theta)d\theta\right\}$$

$$s_{21} + s_{22} + s_{33} = (L_{21} + L_{22} + L_{23}) + (R_{21} + R_{22} + R_{23}) = Re\left\{\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\bar{I}(\theta) - \sum_{k=1}^{3}\frac{(k-1)!}{[\mu e^{-i\tau}]^k}\right]\sec^6\theta(i\cos\theta + 1 + i\sin\theta)d\theta\right\} + Re\left\{2i\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\, e^{i\mu\sin\tau}\sec^6\theta \times (i\cos\theta + 1 + i\sin\theta)d\theta\right\} \tag{59}$$

$$s_4 = L_4 = Re\left\{-\frac{i}{\pi}\int_{-\pi}^{\pi}\frac{6\sec^7\theta}{[\mu e^{-i\tau}]^4}d\theta\right\}$$

$$s_5 = L_5 + R_5 = Re\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[I(\theta) - \sum_{k=1}^{2}\frac{(k-1)!}{[\mu e^{i\tau}]^k}\right] \times \sec^4\theta d\theta\right\} + Re\left\{i\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\, e^{-i\mu\sin\tau}\sec^4\theta d\theta\right\}$$

$$s_6 = L_6 + R_6 = Re\left\{\frac{i}{2\pi}\int_{-\pi}^{\pi}\left[I(\theta) - \sum_{k=1}^{2}\frac{(k-1)!}{[\mu e^{i\tau}]^k}\right] \times \sec^5\theta d\theta\right\} - Re\left\{\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\, e^{-i\mu\sin\tau}\sec^5\theta d\theta\right\}$$

where the term $\bar{I}(\theta)$ in L_{1i} and L_{2i} $(i = 1,2,3)$ and the term

$I(\theta)$ in L_5 and L_6 depend on the value of $\mu = [h^2 + (\xi \cos\theta + \zeta \sin\theta)^2]^{1/2} \sec^2\theta$. Hence, the modulus μ becomes larger when either the integral variable θ approaches $\pm\pi/2$ or the Froude number f is small. Therefore, the computation of the complex integrand in the local terms can best be managed by use of the power series expansion, continued fraction, and asymptotic expansion. Thus, one has

$$\bar{I}(\theta) = \bar{I}_r(\theta) + i\bar{I}_i(\theta) = e^{\zeta}\mathrm{Ei}(-\zeta)|_{\zeta=-\mu e^{-i\tau}} = \begin{cases} e^{-\mu\cos\tau}e^{i\mu\sin\tau}(s_1 - is_2) & \text{for } \mu < 2^{1/2} \\ \text{or} & \\ -\left(\dfrac{1}{\zeta+}\dfrac{1}{1+}\dfrac{1}{\zeta+}\dfrac{2}{1+}\dfrac{2}{\zeta+}\cdots\right)\Big|_{\zeta=-\mu e^{-i\tau}} & \text{for } 2^{1/2} \leqq \mu < 10 \\ \text{or} & \\ \sum_{k=1}^{N}\dfrac{(k-1)!}{(-\zeta)^k}\Big|_{\zeta=-\mu e^{-i\tau}} & \text{for } \mu \geq 10 \end{cases} \tag{60}$$

where

$$s_1 = \gamma + \ln\mu + \sum_{k=1} \frac{\mu^k}{kk!}\cos k\tau$$

$$s_2 = \tau + \sum_{k=1} \frac{\mu^k}{kk!}\sin k\tau$$

and

$$I(\theta) = \bar{I}_r(\theta) - i\bar{I}_i(\theta) = e^{\bar{\zeta}}\mathrm{Ei}(-\bar{\zeta})|_{\bar{\zeta}=-\mu e^{i\tau}} \tag{61}$$

In (59), the local terms (L's) and the radiation terms (R's) are associated with only the real part of the complex integrand; hence by substituting (60) and (61) into (59), one finds

$$L_{11} = \frac{2}{\pi}\int_0^{\pi}\left[\bar{I}_r(\theta) - \sum_{k=1}^{2}\frac{(k-1)!}{\mu^k}\cos k\tau\right]\sec^4\theta d\theta$$

$$L_{12} = \frac{2}{\pi}\int_0^{\pi}\left[\bar{I}_i(\theta) - \sum_{k=1}^{2}\frac{(k-1)!}{\mu^k}\sin k\tau\right]\sec^5\theta d\theta$$

$$L_{13} = \frac{2}{\pi}\int_0^{\pi}\left[\bar{I}_r(\theta) - \sum_{k=1}^{2}\frac{(k-1)!}{\mu^k}\cos k\tau\right]\sec^4\theta \tan\theta d\theta$$

$$L_{21} = -\frac{2}{\pi}\int_0^{\pi}\left[\bar{I}_i(\theta) - \sum_{k=1}^{3}\frac{(k-1)!}{\mu^k}\sin k\tau\right]\sec^5\theta d\theta \tag{62}$$

$$L_{22} = \frac{2}{\pi}\int_0^{\pi}\left[\bar{I}_r(\theta) - \sum_{k=1}^{3}\frac{(k-1)!}{\mu^k}\cos k\tau\right]\sec^6\theta d\theta$$

$$L_{23} = -\frac{2}{\pi}\int_0^{\pi}\left[\bar{I}_i(\theta) - \sum_{k=1}^{3}\frac{(k-1)!}{\mu^k}\sin k\tau\right]\sec^5\theta \tan\theta d\theta$$

$$L_4 = \frac{12}{\pi}\int_0^{\pi}\frac{\sin 4\tau}{\mu^4}\sec^7\theta d\theta$$

$$L_5 = \tfrac{1}{2}L_{11} \qquad L_6 = \tfrac{1}{2}L_{12}$$

and

$$R_{11} = -2\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\sin(\mu\sin\tau)\sec^4\theta d\theta$$

$$R_{12} = 2\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\cos(\mu\sin\tau)\sec^5\theta d\theta$$

$$R_{13} = -2\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\sin(\mu\sin\tau)\sec^4\theta\tan\theta d\theta \tag{63}$$

$$R_{22} = -2\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\sin(\mu\sin\tau)\sec^6\theta d\theta$$

$$R_{23} = -2\int_{-\pi/2}^{\pi/2} e^{-\mu\cos\tau}\cos(\mu\sin\tau)\sec^5\theta\tan\theta d\theta$$

$$R_{21} = -R_{12} \qquad R_5 = -\tfrac{1}{2}R_{11} \qquad R_6 = -\tfrac{1}{2}R_{12}$$

Moreover, by use of the symmetries exhibited by the local terms (62) and the radiation terms (63) at the different quadrants of the $\xi\zeta$ plane, one can show that the integrand of α_{2i}/U with $i = 1,2$ is symmetric with respect to the ξ axis, as might be expected from the physical ground. It follows then that

$$\frac{\alpha_{21}}{U} = -\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} G(\xi,h,\zeta)(L_{11} - R_{11})d\xi d\zeta$$

and (64)

$$\frac{\alpha_{22}}{U} = \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} G(\xi,h,\zeta)(L_{12} - R_{12})d\xi d\zeta$$

where

$$\begin{aligned} G(\xi,h,\zeta) &= G(\mu,\tau) \\ &= 2[(L_{11}+R_{11})(L_{21}-R_{12}) + (r_1 + L_{12} + R_{12}) \times \\ &\quad (r_2 + L_{22} + R_{22}) + (L_{13}+R_{13})(L_{23}+R_{23})] - \\ &\quad (L_{11}+R_{11})L_4 \end{aligned} \tag{65}$$

The first-order and linear second-order wave-resistance and wave-lift coefficients in (53) and (54) can be calculated readily on a computer, using the series expansions of (55). However, the evaluation of the nonlinear contributions represented by (64) cannot be considered as a routine computation. The integrands of (64) are a combination of the local terms (L's) and the radiation terms (R's) that are given as integrals in the variable θ whose exact forms are unknown.

The exponential integrals for complex argument, which constitute the major part of the integrand of the L's in (62), can be computed accurately by making use of information contained in the Handbook of Mathematical Functions.[7]

On the other hand, the integrand of the R's in (63) is the product of an amplitude that decays exponentially (as θ increases) and a trigonometric function having an argument

$$f(\tan\theta) = (\xi + \zeta\tan\theta)(1 + \tan^2\theta)^{1/2}$$

so that the behavior of the oscillation depends not only on θ but also on the ratio ξ/ζ. [In reality, $(\xi/\zeta)^2 = 8$, with $\xi < 0$, corresponds to the cusp line that divides the free surface into the Kelvin's and remaining regions. The wave disturbances are known to be more severe within the Kelvin's region where $(\xi/\zeta)^2 \geqq 8$.] Thus, one has to employ a special integral scheme adaptable to the oscillating and decaying integrand. The detailed account of this scheme is presented in a separate paper.[8]

Finally, to carry out the integration in (64) numerically, it is essential to know how the integrand varies with the distance $r = (\xi^2 + \zeta^2)^{1/2}$ along a line of a constant slope $\tan\beta = \zeta/\xi$, as well as how it changes with the angle β along a circle of a constant radius r. It was confirmed by numerical trial that, at most, the calculation of integrands at the following intersections of the concentric circles and the straight lines was required: 1) the circles of radius $0 < r \leqq 3.0$ with the increment 0.1, and 2) the lines making angles $0° < \beta \leqq$

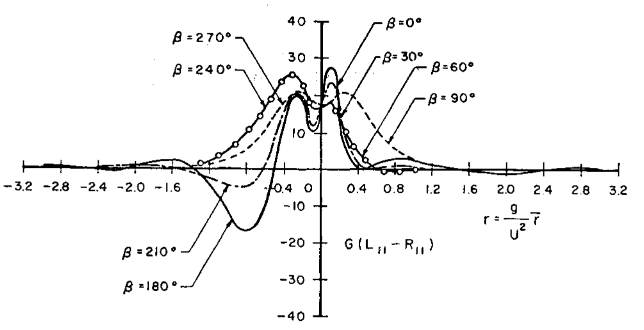

Fig. 2 Nonlinear contribution to resistance from various points on free surface.

90° with the increment 5°. Thus, from (64) one has the approximations

$$\frac{\alpha_{21}}{U} \sim -\frac{1}{2\pi}\left[\int_0^{\pi/2}\int_0^3 G(r,\beta;h)(L_{11} - R_{11})rdrd\beta + \int_\pi^{3\pi/2}\int_0^3 G(r,\beta;h)(L_{11} - R_{11})rdrd\beta\right]$$

and (66)

$$\frac{\alpha_{22}}{U} \sim \frac{1}{2\pi}\left[\int_0^{\pi/2}\int_0^3 G(r,\beta;h)(L_{12} - R_{12})rdrd\beta + \int_\pi^{3\pi/2}\int_0^3 G(r,\beta;h)(L_{12} - R_{12})rdrd\beta\right]$$

Results

The computed results indicate that, in the three-dimensional water-wave problem of a submerged body in arbitrary motion, it is more important to correct the wave-induced forces for the nonlinear second-order contribution by the presence of the first-order waves on the free surface, than it is to correct for the linear second-order contribution resulting from the modification of flow around the body by the effect of the free surface.

In Fig. 1, for example, one may note the variation of the dimensionless pressure-slope over the free surface when the Froude number $f = U/(gh)^{1/2}$ is unity. The dimensionless pressure-slope is the rate of change of the pressure in the direction of the motion of a submerged sphere; it is defined by $G = p_\xi/M^2\rho$ where $M = -Ua^3/2$ [see Eq. (65)]. The pressure-slope decays quickly in all directions, and hence the pressure itself, caused by the first-order waves, tends to be constant.

In Figs. 2 and 3, nonlinear contributions to resistance $G(L_{11} - R_{11})$ and to lift $G(L_{12} - R_{12})$ from various points on the free surface are presented for $f = 1.0$.

In passing, one should note that the coefficients α_{21}/U and α_{22}/U which are obtained by integrating $G(L_{11} - R_{11})$ and $G(L_{12} - R_{12})$ over the entire free surface are physically the horizontal and vertical rates of change of the velocity $\partial\varphi^{(p)}/\partial x$ induced by the pressure. The significant variations of the quantities $G(L_{11} - R_{11})$ and $G(L_{12} - R_{12})$ are confined within the small region where $r \leqq 3.0$, so that, in the process of the integration for α_{2i}/U, $i = 1,2$, one need not be concerned about curtailing the infinite range of the integration.

The p_{1i}, p_{2i}, and α_{2i}/U in Eqs. (53) and (54) are dependent on the Froude number alone, and once these quantities are computed for the different values of f (see Table 1), the wave-resistance coefficient C_R and the wave-lift coefficient C_L of a submerged sphere corresponding to any radius-submergence ratio $c = a/h$ can readily be obtained.

In Figs. 4 and 5, one plots: 1) the first approximations obtained by the first term in Eqs. (53) and (54) only, 2) the second approximations consisting of the first two terms, and 3) the complete second-order forces for the case $a/h = \frac{1}{2}$. The curves of 1 and 2 almost coincide in both figures; hence the linear second-order correction is insignificant. However, the difference between curve 1 and curve 3 is remarkable. In Fig. 4, curves 1 and 3 cross over at $f = 6.5$, and for $f < 6.5$ the complete second-order theory yields the higher resistance while for $f > 6.5$ this theory gives the lower estimate compared with the result based on the linear first-order theory. In Fig. 5, curves 1 and 3 intersect at $f = 5.2$, and for $f < 5.2$ the lift due to the complex second-order theory is higher, although for $f > 5.2$ this theory yields the lower estimate compared with the result based on the linear first-order theory.

It is interesting to see that in this problem the over-all difference between the results of linear first-order theory and

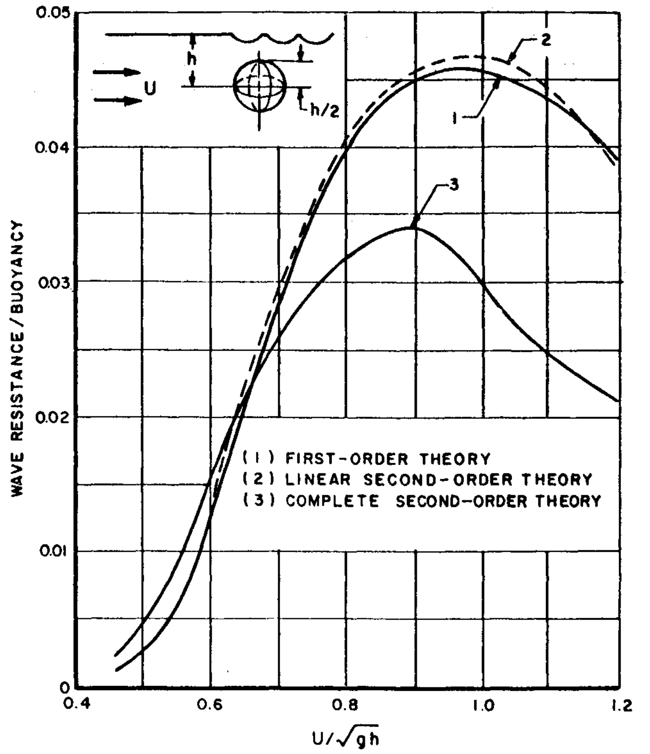

Fig. 4 Wave-resistance coefficients.

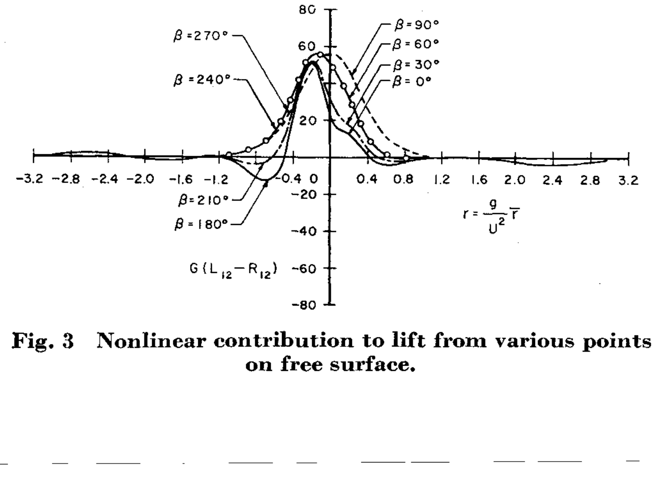

Fig. 3 Nonlinear contribution to lift from various points on free surface.

Table 1 Factors in the wave-resistance and wave-lift coefficient

f	p_{11}	p_{21}	p_{12}	$2\pi\alpha_{21}/U$	$2\pi\alpha_{22}/U$
0.5	−0.0038	0.0018	0.0005	0.0005	0.0001
0.6	−0.0148	0.0087	0.0067	0.0049	−0.0075
0.7	−0.0353	0.0190	0.0355	−0.0523	−0.1231
0.8	−0.0532	0.0111	0.1115	−0.6042	−0.1149
0.9	−0.0435	−0.0630	0.2568	−3.5208	−4.7530
1.0	0.0253	−0.2784	0.4870	−17.8140	−17.6078
1.2	0.4897	−1.6359	1.2452	−164.9394	−133.9681

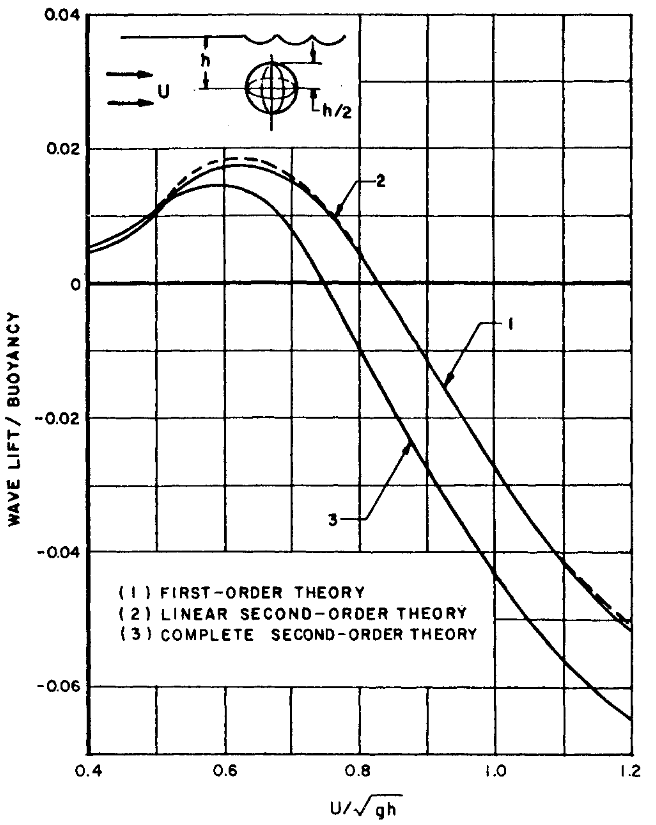

Fig. 5 Wave-lift coefficients.

the results of complete second-order theory is not so severe as the case of the two-dimensional problem of a circular cylinder treated by Tuck.[1] This is attributable to the fact that in the three-dimensional case most of the wave disturbances traveling rearward from the submerged body are confined with the Kelvin's angle, while disturbances in the two-dimensional problem are considered to propagate throughout the entire rear half-plane.

One must not misinterpret the present results, based as they are on the complete second-order theory; they are not the ultimate answer to the problem of wave-induced forces acting on a sphere in a steady translation, since the exact potential theory is rather a poor model for the real-fluid problem that is dominated by viscous effects. Nevertheless, these results are meaningful; they demonstrate that in water-wave problems the inclusion of the nonlinear free-surface effect is far more important than the consideration of any linear contributions.

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